Period 4, Nov 21, 2024

## Definition of the Exponential Function

The **exponential function $f$ with base $b$** is defined by

$$f(x) = b^x \quad \text{or} \quad y = b^x,$$

where $b$ is a positive constant other than 1 ($b > 0$ and $b \neq 1$) and $x$ is any real number.

$y = 3^x$

| $x$ | $y$ | |
|---|---|---|
| 0 | 1 | $= 3^0$ |
| 1 | 3 | $= 3^1$ |
| 2 | 9 | $= 3^2$ |
| 3 | 27 | $= 3^3$ |
| $-1$ | $\frac{1}{3}$ | $= 3^{-1}$ |
| $-2$ | $\frac{1}{3^2} = \frac{1}{9}$ | $= 3^{-2}$ |

## Characteristics of Exponential Functions of the Form $f(x) = b^x$

1. The domain of $f(x) = b^x$ consists of all real numbers: $(-\infty, \infty)$. The range of $f(x) = b^x$ consists of all positive real numbers: $(0, \infty)$.
2. The graphs of all exponential functions of the form $f(x) = b^x$ pass through the point $(0, 1)$ because $f(0) = b^0 = 1 (b \neq 0)$. The $y$-intercept is 1. There is no $x$-intercept.
3. If $b > 1$, $f(x) = b^x$ has a graph that goes up to the right and is an increasing function. The greater the value of $b$, the steeper the increase.
4. If $0 < b < 1$, $f(x) = b^x$ has a graph that goes down to the right and is a decreasing function. The smaller the value of $b$, the steeper the decrease.
5. $f(x) = b^x$ is one-to-one and has an inverse that is a function.
6. The graph of $f(x) = b^x$ approaches, but does not touch, the $x$-axis. The $x$-axis, or $y = 0$, is a horizontal asymptote.

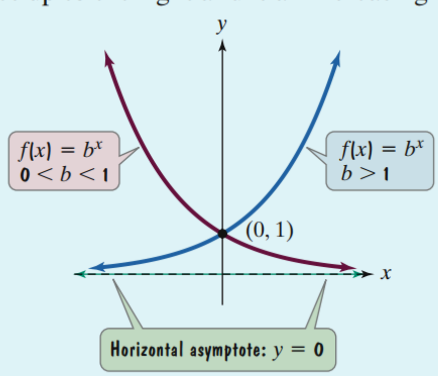
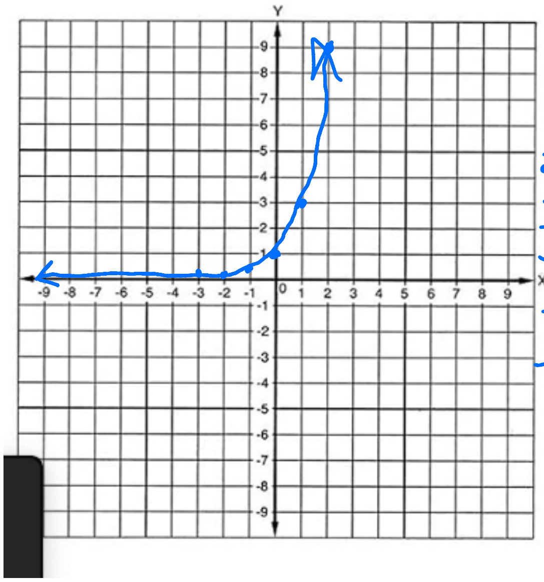



**Table 3.1 Transformations Involving Exponential Functions**

In each case, $c$ represents a positive real number.

| Transformation | Equation | Description |
|---|---|---|
| Vertical translation | $g(x) = b^x + c$<br>$g(x) = b^x - c$ | • Shifts the graph of $f(x) = b^x$ upward $c$ units.<br>• Shifts the graph of $f(x) = b^x$ downward $c$ units. |
| Horizontal translation | $g(x) = b^{x+c}$<br>$g(x) = b^{x-c}$ | • Shifts the graph of $f(x) = b^x$ to the left $c$ units.<br>• Shifts the graph of $f(x) = b^x$ to the right $c$ units. |
| Reflection | $g(x) = -b^x$<br>$g(x) = b^{-x}$ | • Reflects the graph of $f(x) = b^x$ about the $x$-axis.<br>• Reflects the graph of $f(x) = b^x$ about the $y$-axis. |
| Vertical stretching or shrinking | $g(x) = cb^x$ | • Vertically stretches the graph of $f(x) = b^x$ if $c > 1$.<br>• Vertically shrinks the graph of $f(x) = b^x$ if $0 < c < 1$. |
| Horizontal stretching or shrinking | $g(x) = b^{cx}$ | • Horizontally shrinks the graph of $f(x) = b^x$ if $c > 1$.<br>• Horizontally stretches the graph of $f(x) = b^x$ if $0 < c < 1$. |

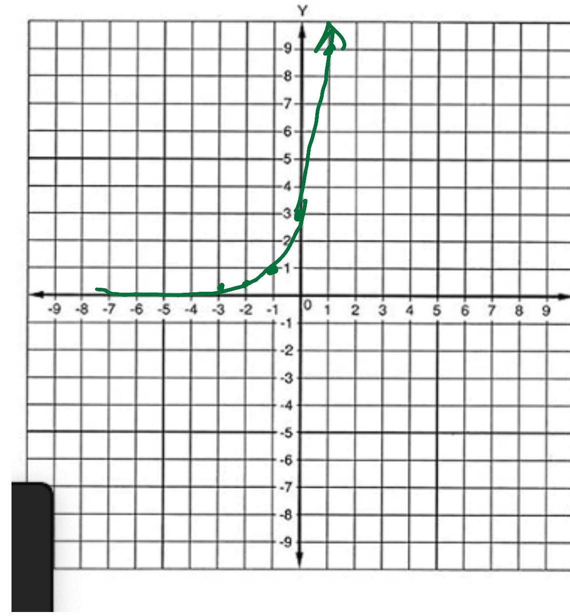
graph of $f(x) = 3^x$ t



| x | y |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 9 |
| 3 | 27 |
| -1 | $\frac{1}{3}$ |
| -2 | $\frac{1}{9}$ |
| -3 | $\frac{1}{27}$ |
| -4 | $\frac{1}{81}$ |

$g(x) = 3^{x+1}$.

move Left 1



| x | y |
|---|---|
| -1 | $1 = 3^0$ |
| 0 | 3 |
| 1 | 9 |
| 2 | 27 |
| 3 | 81 |
| -2 | $\frac{1}{3}$ |
| -3 | $\frac{1}{9}$ |

**natural base.**

**natural exponential function.**

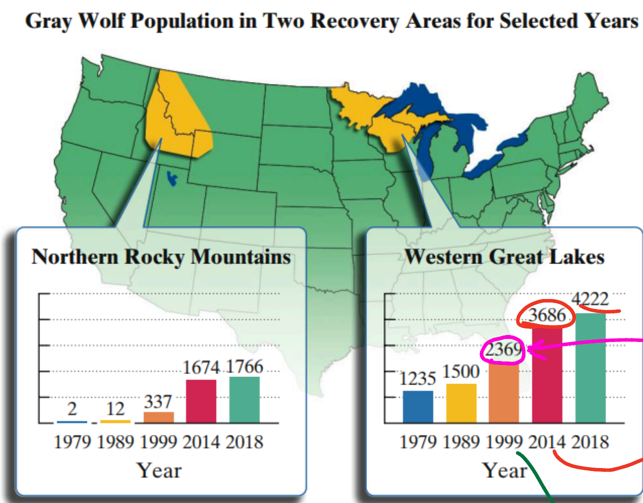
**Table 3.2**

| $n$ | $\left(1+\frac{1}{n}\right)^n$ |
|---|---|
| 1 | 2 |
| 2 | 2.25 |
| 5 | 2.48832 |
| 10 | 2.59374246 |
| 100 | 2.704813829 |
| 1000 | 2.716923932 |
| 10,000 | 2.718145927 |
| 100,000 | 2.718268237 |
| 1,000,000 | 2.718280469 |
| 1,000,000,000 | 2.718281827 |

$y = e^{ax} + b$

## EXAMPLE 6 Gray Wolf Population

Insatiable killer. That's the reputation the gray wolf acquired in the United States in the nineteenth and early twentieth centuries. Although the label was undeserved, an estimated two million wolves were shot, trapped, or poisoned. By 1960, the population was reduced to 800 wolves. **Figure 3.6** shows the rebounding population in two recovery areas after the gray wolf was declared an endangered species and received federal protection.




Really close

$x=36$ $y=3686$

21 years after 1978

$x=21$

**Figure 3.6**
*Source:* U.S. Fish and Wildlife Service

The exponential function

$$f(x) = 1144e^{0.0325x}$$

$F(21) = 1144e^{0.0325(21)} = 2263.7$

models the gray wolf population of the Western Great Lakes, $f(x)$, $x$ years after 1978.

**a.** According to the model, what was the gray wolf population, rounded to the nearest whole number, of the Western Great Lakes in 2018? $x = 40$

**b.** Does the model underestimate or overestimate the gray wolf population of the Western Great Lakes in 2018? By how much?

$Y = 1144e^{0.0325(40)}$

4198

$4222 - 4198 = 24$

## Compound Interest

**principal**, $P$,

**Table 3.3**

| Time in Years | Accumulated Value after Each Compounding |
|---|---|
| 0 | $A = P$ $= 2000$ |
| 1 | $A = P(1 + r)$ $= 2120$ |
| 2 | $A = P(1 + r)(1 + r) = P(1 + r)^2$ $= 2247.20$ |
| 3 | $A = P(1 + r)^2(1 + r) = P(1 + r)^3$ $= 2382.03$ |
| 4 | $A = P(1 + r)^3(1 + r) = P(1 + r)^4$ |
| Retire after 38 | ⋮ $2000(1.06)^{38} = 18{,}308.5$ |
| $t$ | $A = P(1 + r)^t$ |

\$2,000.00

what is 6% of \$2000?

$(0.06)(2000) = 120$

$(0.06)(2120) = 127.2$

$127.2 + 2120 = 2247.2$

$2000(1+0.06)^1 = 2120$

$2000(1.06)^2 = 2247.2$

$2000(1.06)^3 = 2000(1.191) = 2382.03$

**Table 3.4 Interest Plans**

| Name | Number of Compounding Periods per Year | Length of Each Compounding Period |
|---|---|---|
| Semiannual Compounding | $n = 2$ | 6 months |
| Quarterly Compounding | $n = 4$ | 3 months |
| Monthly Compounding | $n = 12$ | 1 month |
| Daily Compounding | $n = 365$ | 1 day |

Years 38

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

$$2000\left(1+\frac{0.06}{2}\right)^{38\cdot 2} = 2000(1.03)^{76} = 18{,}908.58$$

$$2000\left(1+\frac{0.06}{4}\right)^{38\cdot 4} = 2000(1.015)^{152} = 19225.09$$

$$2000\left(1+\frac{0.06}{365}\right)^{38\cdot 365} = 2000(1.000164384)^{13870} = 19549.69$$

### Formulas for Compound Interest

After $t$ years, the balance, $A$, in an account with principal $P$ and annual interest rate $r$ (in decimal form) is given by the following formulas:

1. For $n$ compounding periods per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$. $= 2000 \cdot e^{(0.06)(38)} = 2000e^{2.28} = 2000(9.77668)$ $= 19353.36$

$2,000 at 11% for 38 years compounding continuously

$2000e^{(0.11)(38)} = 2000e^{4.18} = 2000(65.4) = 130,731.71$

**EXAMPLE 7** **Choosing between Investments**

You decide to invest $8000 for 6 years and you have a choice between two accounts. The first pays 1.20% per year, compounded monthly. The second pays 1.19% per year, compounded continuously. Which is the better investment?

$T = 6$ years

$R = 1.2\% = 0.012$

$n = 12$

$P = 8000$

$$A = 8000\left(1 + \frac{0.012}{12}\right)^{12 \cdot 6} = 8000(1.001)^{72} = 8000(1.0746168) = 8596.933$$

$$A = Pe^{rT} = 8000e^{(0.0119)6} = 8000e^{0.0714} = 8000(1.074) = 8592.086$$

Follow the seven step strategy to graph the following rational function.

$$f(x) = \frac{x-3}{x^2-9} = \frac{(x-3)}{(x+3)(x-3)}$$

$$F(-x) = \frac{(-x)-3}{(-x)^2-9} = \frac{-x-3}{x^2-9}$$

$$-F(x) = \frac{-1}{1}\left(\frac{x-3}{x^2-9}\right) = \frac{-1(x-3)}{1(x^2-9)} = \frac{-x+3}{x^2-9}$$

$F(x) \neq -F(x)$, $F(-x) \neq F(x)$ } No symmetry

To graph the function, first determine the symmetry of the graph of f. Choose the correct answer below.

None

What is the y-intercept? Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

Set $x=0$ $\quad F(0) = \frac{0-3}{0^2-9} = \frac{-3}{-9} = \frac{1}{3}$

What is/are the x-intercept(s)? Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

Set $y=0$ $\quad 0 = \frac{x-3}{(x^2-9)} = \frac{\cancel{x-3}}{(\cancel{x-3})(x+3)} \Rightarrow \frac{1}{\cancel{x+3}} = 0 \; (x+3)$

$x=3 \Rightarrow F(3) = \frac{0}{0} = \emptyset$

$1 = 0$ No gonna work

$\boxed{\text{No x-intercepts}}$

Find the vertical asymptote(s) or hole(s). Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

$F(x) = \frac{(\cancel{x-3})}{(\cancel{x-3})(x+3)}$ $\quad x=3$ hole $\quad \boxed{x=-3 \Rightarrow \text{vertical asymptote}}$

Find the horizontal asymptote(s). Select the correct choice below and, if necessary, fill in the answer box to complete your choice.

$x \to \infty$

$x \to -\infty$

$F(x) = \frac{(\cancel{x-3})}{(\cancel{x-3})(x+3)} = \frac{1}{x+3}$ $\quad x \neq 3$

$\frac{1}{\infty} = 0$

$\frac{1}{-\infty} = 0$

$\boxed{\text{horizontal asymptote } y=0}$

Plot points between and beyond each x-intercept and vertical asymptote. Find the value of the function at the given value of x.

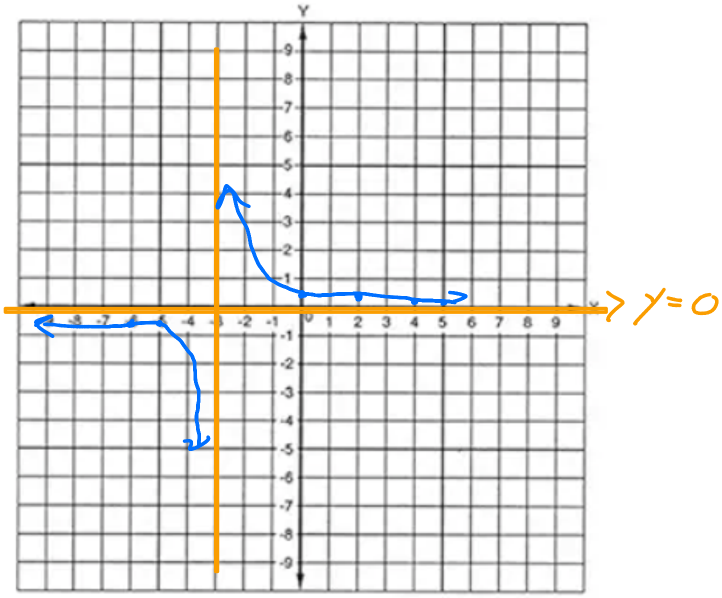
| x | −6 | −5 | 2 | 4 | 5 |
|---|---|---|---|---|---|
| | $-\frac{1}{3}$ | $-\frac{1}{2}$ | $\frac{1}{5}$ | $\frac{1}{7}$ | $\frac{1}{8}$ |

$F(x) = \frac{(x-3)}{(x-3)(x+3)}$ $\quad \underbrace{x \neq 3 \text{ or } -3}$

$\frac{(-6-3)}{(-6-3)(-6+3)} = \frac{\cancel{-9}}{(\cancel{-9})(-3)}$

$\frac{(-5-3)}{(-5-3)(-5+3)} = \frac{(\cancel{-8})}{(\cancel{-8})(-2)}$

$\frac{1}{x+3}$ $\quad \frac{1}{2+3}$




Find the vertical asymptotes, if any, and the values of x corresponding to holes, if any, of the graph of the rational function.

$$f(x) = \frac{x-2}{x^2-8x+12}$$

$x^2-8x+12 = (x-2)(x-6)$

$$F(x) = \frac{(x-\cancel{2})}{(x-\cancel{2})(x-6)}$$

$x=2$ hole

$x=6$ vertical asymptote

$$f(x) = \frac{6x^2}{x^2-4}$$

$$F(-x) = \frac{6(-x)^2}{(-x)^2-4} = \frac{6x^2}{x^2-4}$$

$$F(x) = \frac{6x^2}{(x-2)(x+2)}$$

vertical asy

$x=2, -2$

$F(-x) = F(x)$

Even

$y$-axis symmetry

The equation for f is given by the simplified expression that results after performing the indicated operation. Write the equation for f and then graph the function.

$$\frac{1-\frac{4}{x+3}}{1+\frac{2}{x-3}} = \frac{\frac{x+3}{x+3}-\frac{4}{x+3}}{\frac{x-3}{x-3}+\frac{2}{x-3}} = \frac{\frac{x+3-4}{x+3}}{\frac{x-3+2}{x-3}} = \frac{\frac{x-1}{x+3}}{\frac{x-1}{x-3}} = \frac{\cancel{(x-1)}(x-3)}{(x+3)\cancel{(x-1)}} = \frac{x-3}{x+3}$$

$x \neq 3$ or $-3$

$x \neq 1$

Vertical asy $x=-3$

hole $x=1$

Horizontal asy $y=1$

$x \to \infty$ $\quad \frac{x-3}{x+3} \approx \frac{x}{x} = 1$

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$$F(x) = \frac{x^2-25}{x}$$

$$F(-x) = \frac{(-x)^2-25}{-x} = \frac{x^2-25}{-x} = -\frac{x^2-25}{x}$$

$$\frac{-x^2+25}{x}$$

$F(x) \neq F(-x)$

Nope = Not Even

$$-F(x) = -\left(\frac{x^2-25}{x}\right) = \frac{-x^2+25}{x}$$

$F(-x) = -F(x)$

odd